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# Savings and growth in neoclassical growth models: A comment on "Is Piketty's "second law of capitalism" fundamental?"

Fracisco Alvarez-Cuadrado[1]

*Department of Economics, McGill University, Montreal H3A 2T7, Canada*

## HIGHLIGHTS

- This paper explores the theoretical response of saving rates to the rate of income growth.
- This response depends on the consumption intertemporal elasticity of substitution.
- We provide some empirical evidence on the sign of this response.
- We reassess some of the claims in Krusell and Smith (2015) which turn out to be true only under very specific circumstances.

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## ABSTRACT

This note explores the response of saving rates to the rate of income growth in models of optimal growth. Contrary to the claims in Krusell and Smith (2015) the sign of this response depends on the consumption intertemporal elasticity of substitution.



## 1. Introduction

In a very interesting review essay, Krusell and Smith (2015) make a claim on the response of the steady state saving rate to changes in the rate of income growth, $g$, along the balanced growth path of a standard neoclassical growth model of optimal savings: "Optimal-savings theory implies, more generally, that on a balanced growth path, both the gross and the net saving rates depend positively on $g$" (p. 736). This claim does not seem to be either theoretically correct or empirically probable in view of the existing evidence on the structural parameters of the model, particularly on the consumption elasticity of intertemporal substitution.

The stated goal of the paper, clearly summarized by its title, is to explore the implications of Piketty's (2014) "second fundamental law of capitalism". This law states that under a constant net saving rate, $\tilde{s}$, the steady state capital-to-(net) income ratio, $k/\tilde{y}$, is given by $\tilde{s}/g$. As the authors convincingly argue, the theory of saving embedded in this second law is highly implausible, since it implies that the gross saving rate tends to one as the rate of income growth tends to zero. To see this, notice that since $k/\tilde{y} = \tilde{s}/g$, then $k/\tilde{y}$ tends to infinity as $g$ tends to zero, but since $k/\tilde{y} = k/(y - \delta k)$, where $y$ is gross output and $\delta$ the rate of depreciation, this implies that $\delta k$ tends to $y$, and therefore all output is devoted to replacement investment, with nothing being left for consumption. I believe this is the main message of their paper, which I find very compelling. Then the authors proceed with an exercise aimed at assessing the empirical relevance of different theories of saving. For this purpose, they combine long-run data on saving and growth rates for several countries and conclude that both the net and gross saving rate tend to increase with $g$. Then they explore the theoretical response of saving rates to the rate of income growth along the balanced growth path of alternative models, particularly a model with a constant net saving rate –Piketty's model – one with a constant gross saving rate –the textbook Solow model – and a model of optimal savings. They conclude that their evidence on saving and growth rates "is at least qualitatively consistent with the optimizing model" (p. 739) since "in the usual optimizing growth model with standard production function, both saving rates (net and gross) are increasing in $g$" (p. 737).

The purpose of this short note is threefold. First, I will show that their last theoretical claim is not, in general, true. Second, I

*E-mail address:* francisco.alvarez-cuadrado@mcgill.ca.

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will explore which restrictions on the parameters of the model make the claim true and, tentatively, review some of the existing evidence on the empirical plausibility of these restrictions. Finally, I will use the data in Krusell and Smith (2015) to assess the response of saving rates to the two sources of income growth along the balanced growth path of neoclassical models: labor-augmenting technological change and population growth.

## 2. Savings and growth in an optimal growth model

Consider a continuous-time version of the standard neoclassical growth model under optimal savings, see for instance Barro and Sala-i Martin (1999, Chapter 2). The dynamic behavior of this model is given by the following pair of differential equations on consumption, $c(t)$, and capital, $k(t)$, per unit of effective labor,

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\sigma}\left(f'(k(t)) - \delta - \rho - \sigma g\right),$$
$$\dot{k}(t) = f(k(t)) - c(t) - (\delta + n + g)\,k(t),$$

together with an initial and a final (transversality) conditions, where $\rho$ is the rate of time preference, $n$ is the rate of growth of the labor force, and $f(k(t))$ is a neoclassical production function with the standard properties. Although, Piketty (2014) denotes by $g$ the (long run) rate of growth of national income, which combines both increases in the labor force and in technology, Krusell and Smith (2015) seems to abstract from population growth and, in their analysis of optimal savings, $g$ stands for the rate of growth of labor-augmenting technology, see their footnote 14. I will follow this last convention. Finally, $\sigma > 0$ is the inverse of the elasticity of intertemporal substitution, assumed to be constant and an additional restriction, $\rho - n > (1-\sigma)\,g$, is required to ensure that lifetime utility remains bounded along a balanced growth path. It is worth noticing that Krusell and Smith (2015) focus on the specific case in which $\sigma = 1$, and, as we will see, this turns out to have non trivial implications.

As they do, I restrict the analysis to the steady state of the system, which corresponds to a balanced growth path in per capita or aggregate terms.[2] I denote this steady state using asterisks. Since $f()$ is concave, $f'()$ is monotonic and hence the steady state capital stock, $k^* = f'^{-1}(\delta + \rho + \sigma g)$, decreases in the rate of technical change, $\frac{\partial k^*}{\partial g} < 0$. It is straight forward to derive the steady state saving rates, both gross, $s$, and net, $\tilde{s}$, as

$$s^* \equiv \frac{f(k^*) - c^*}{f(k^*)} = (\delta + n + g)\frac{k^*}{f(k^*)} = \alpha\left(k^*\right)\frac{(\delta + n + g)}{\rho + \delta + \sigma g}, \quad (1)$$

$$\tilde{s}^* \equiv \frac{f(k^*) - \delta k^* - c^*}{f(k^*) - \delta k^*} = \frac{(n+g)\,k^*}{f(k^*) - \delta k^*}$$
$$= \alpha\left(k^*\right)\frac{(n+g)}{\rho + (1 - \alpha(k^*))\,\delta + \sigma g}, \quad (2)$$

where $\alpha(k^*)$ is the steady state elasticity of output to capital.[3]

At this level of generality, the conditions on the response of the saving rate to the rate of technical change are difficult to interpret. Nonetheless, under a Cobb–Douglas technology, $y = k^\alpha$, like the one used in Figure 1 in Krusell and Smith (2015), these responses are given by[4]

$$\frac{\partial s^*}{\partial g} = \frac{\alpha\left(\rho + (1-\sigma)\,\delta - \sigma n\right)}{(\rho + \delta + \sigma g)^2}, \quad (3)$$

and

$$\frac{\partial \tilde{s}^*}{\partial g} = \frac{\alpha\left(\rho + (1-\alpha)\,\delta - \sigma n\right)}{(\rho + (1-\alpha)\,\delta + \sigma g)^2}, \quad (4)$$

and their signs depend on the size of the elasticity of intertemporal substitution, $1/\sigma$.[5]

The gross saving rate increases in $g$ as long as

$$\frac{1}{\sigma} > \frac{(\delta + n)}{\rho + \delta} \quad (5)$$

is satisfied. Similarly, the net saving rate increases in $g$ as long as

$$\frac{1}{\sigma} > \frac{n}{\rho + (1-\alpha)\,\delta}. \quad (6)$$

There are several things to notice. First, it is clear that if the first inequality holds, so does the second. In this sense, if the response of the gross saving rate is positive so is the response of the net saving rate. Second, in the absence of population growth, $n = 0$, the net saving rate is always increasing in $g$. Third, under logarithmic preferences, $\sigma = 1$, both saving rates increase in $g$, since in this case bounded lifetime utility requires $\rho > n$. This case, which by no means is general, underlies the analysis in Krusell and Smith (2015) and, I believe, drives their conclusion about the positive response of the saving rate to $g$ in the model of optimal savings. Fourth, in general, the behavior of the steady state saving rates is ambiguous because it involves offsetting income and substitution effects. On the one hand, an increase in $g$ lowers steady state capital, $k^*$, increasing the rate of return on saving, $f'(k^*) = \alpha\,(k^*)^{\alpha-1}$. This substitution effect tends to increase the saving rate. On the other hand, an increase in $g$ permanently increases the growth rate of the economy. Households anticipating higher future income tend to increase their consumption. This income effect tends to lower the saving rate. When the substitution elasticity, $1/\sigma$, is large enough, the substitution effect dominates the income effect and the steady state saving rates increase with the rate of per capita income growth, $g$.[6] Similar restrictions on the elasticity of intertemporal substitution can be derived for alternative specifications of the production function. Focusing in the case of the gross saving rate, the threshold for the production function introduced in Section 2 of Krusell and Smith (2015), $y = k^\alpha + \delta k$, is given by $1/\sigma > (\delta + n)/(\rho + \alpha\delta)$, for a technology linear in capital, $y = Ak$, that allows for endogenous growth, the threshold coincides with (5), and for a Constant Elasticity of Substitution technology, $y = \left(\alpha k^{\frac{\varepsilon-1}{\varepsilon}} + (1-\alpha)\right)^{\frac{\varepsilon}{\varepsilon-1}}$, the threshold is given by $1/\sigma > (\varepsilon(\delta + n) + (\varepsilon - 1)\,g)/(\rho + \delta)$, where $\varepsilon$ is the elasticity of substitution between capital and labor.[7]

Fig. 1 depicts the elasticities of both saving rates to the rate of labor-augmenting technical change, $g$, for different values of the elasticity of intertemporal substitution. As Fig. 1 in Krusell and Smith (2015) I assume a Cobb–Douglas technology with an

[2] Carroll et al. (2000) explore the relation between savings and growth along the transitional path of a model of endogenous growth.

[3] Notice that as long as steady state consumption is positive the net saving rate is below the gross saving rate, since $\tilde{s} = (s - \delta k/y)/(1 - \delta k/y)$. Furthermore, steady state net savings is zero only if $n + g = 0$.

[4] Appendix 2C in Barro and Sala-i Martin (1999) derives a version of Eq. (1) for a Cobb–Douglas production function where the elasticity of output to capital is constant, $\alpha(k^*) = \alpha$.

[5] Of course, the signs of these responses also depend on other parameters of the model: $\delta$, $n$ and $\rho$. Nonetheless, I believe it is useful to express the restrictions in terms of the elasticity of intertemporal substitution, since it seems to be a broader agreement on the values of these other parameters. Toda (2018) and Carroll and Young (2018) also show that the links between saving, inequality, and growth are not invariant to the elasticity of intertemporal substitution.

[6] Finally, it is worth noticing that in the absence of population growth, $n = 0$, as $\rho \to 0$ the threshold defined by (5) tends to one. In this limit, we recover a familiar result under logarithmic preferences, $\sigma = 1$, where income and substitution effects cancel out and the steady state saving rate is independent of the rate of technological change. Nonetheless, this limit case is problematic, since in the absence of discounting, the objective function, $\int_0^\infty u\left(C_i(t)\right)e^{-(\rho-n)t}dt$ where $C_i$ is consumption per capita, is unbounded and the problem is not well defined, since it violates the restriction, $\rho - n > (1-\sigma)\,g$.

[7] In the case of a linear technology, we set $g = 0$ and $k$ denotes per capita or aggregate capital. The rate of output growth along the stable growth path of this model is an increasing function of $A$.

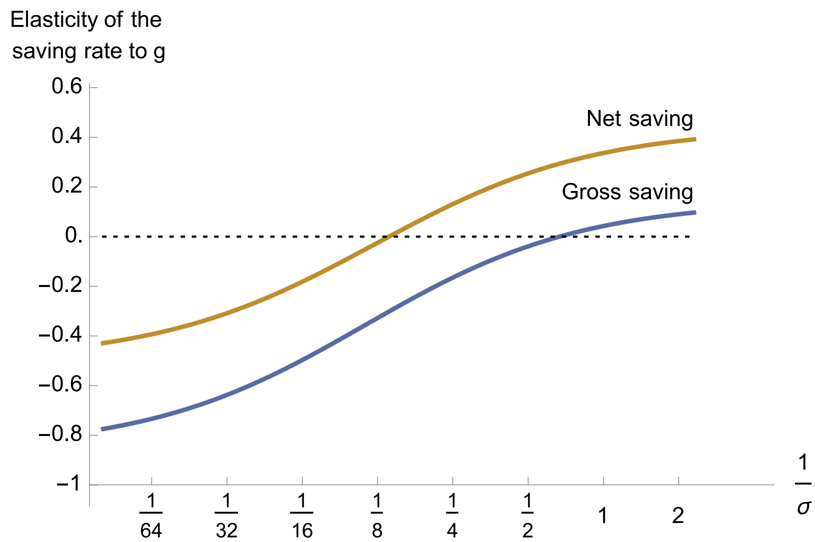


**Fig. 1.** Elasticity of the saving rate to $g$, under Cobb–Douglas technology. Parameter values: $\alpha = 0.36$, $\delta = 0.05$, $\rho = 0.04$, and $n = g = 0.01$.

elasticity of output to capital $\alpha = 0.36$, a discount factor $\rho = 0.04$, and a rate of depreciation $\delta = 0.05$. Additionally, I set the rates of population growth and labor-augmenting technical change to one percent, $n = g = 0.01$. For this specific parameter values, the gross saving rate increases on the rate of technical change when the elasticity of substitution exceeds 2/3.

Although most aggregate models calibrated to match business cycle facts use values of $1/\sigma$ that tend to exceed the threshold in (5), see Guvenen (2006) for a nice summary, most econometric estimates of the elasticity of intertemporal substitution tend to lie well below this threshold. For instance, Hall (1988) concludes that the elasticity is not above 1/5, Mankiw and Zeldes (1991) reports an estimate of 1/6, Ogaki and Reinhart (1998) reports estimates in the range of 1/3, Vissing-Jorgensen (2002) reports an estimate of 1/3 for stockholders and of 1 for bondholders, Yogo (2004) finds that this elasticity is small and not significantly different from 0, and, recently, Cashin and Unayama (2016), reports an estimate of 1/5. Given this evidence, it seems unwarranted, even on empirical grounds, to conclude that the positive dependence of saving and growth rates is "a natural outcome of standard theories of saving based on optimizing behavior and widely used in macroeconomics" (Krusell and Smith, 2015, p. 727).

## 3. A quick look at the data

Finally, in the optimizing model the response of saving rates to the rate of income growth, $g + n$, depends on whether income growth is driven by technical change, $g$, or population growth, $n$. While the response of saving rates to $g$ depends on the size of the elasticity of intertemporal substitution, its response to $n$ is always positive, for instance under a Cobb–Douglas technology these responses are given by $\partial s^*/\partial n = \alpha/(\rho + \delta + \sigma g) > 0$ and $\partial \tilde{s}^*/\partial n = \alpha/(\rho + (1-\alpha)\,\delta + \sigma g) > 0$. This contrasts with the implications of models of exogenous saving, such as the textbook Solow model or Piketty's model, where both sources of growth, $g$ and $n$, have the same effect on steady state saving rates. This difference suggests a natural test for the theories of savings embedded in each of the three models along the lines of that in Krusell and Smith (2015). Using data for a panel of countries in Piketty and Zucman (2014), these authors regress 20-year averages of saving rates on growth rates and country dummies. Since this dataset includes measures of population, I decompose the rate of income growth into population growth and technical change. The first two columns of Table 1 reproduce the results in Krusell and Smith (2015). For the gross saving rate, the coefficient on the rate of income growth, $g + n$, is positive, 0.013, and statistically significant.[8] That for the net saving rate is 0.019 and statistically significant. The last two columns of Table 1 report results that decompose income growth into its two elements. For the gross saving rate, the coefficient on the rate of technical change is no longer statistically different from zero, while that on population growth is positive and significant. For the net saving rate, the coefficient on the rate of technical change remains positive but it is statistically significant only at the 10% level, while that on population growth is positive and significant at the 1% level.[9]

**Table 1**
Saving rates and growth.

| | Gross saving rate | | Net saving rate | |
|---|---|---|---|---|
| Constant | 0.162*** | 0.160*** | 0.053*** | 0.049*** |
| | (0.016) | (0.016) | (0.0167) | (0.0145) |
| Income growth, g + n | 0.013*** | | 0.019*** | |
| | (0.006) | | (0.006) | |
| Technical change, g | | 0.009 | | 0.011* |
| | | (0.006) | | (0.006) |
| Population growth, n | | 0.038** | | 0.065*** |
| | | (0.018) | | (0.017) |
| $R^2$ | 0.78 | 0.79 | 0.50 | 0.63 |
| Observations | 34 | 34 | 34 | 34 |

Note: Data from Piketty and Zucman (2014) for France (1831–2010), United States and United Kingdom (1871–2010), Germany (1951–2010), and Australia, Canada, Italy, and Japan (1971–2010). All regressions include country dummies. Standard errors are reported in parentheses.
*Significant at 1%.
**Significant at 5%.
***Significant at 1%.

Although these results are only suggestive, we can explore their implications for the three models of saving.[10] In Piketty's model,

8 These estimates are in line with those reported by Krusell and Smith (2015). For the gross saving rate they report a coefficient of 0.018 and for the net saving rate one of 0.024.

9 Using data from the Bureau of Economic Analysis, Krusell and Smith (2015) conduct a second exercise that uses 10-year averages for the US. Reproducing a version of their exercise that decomposes income growth into population growth and technical change conveys a similar picture than the results reported in Table 1.

10 As Krusell and Smith (2015) acknowledge it is difficult to make long-run evaluations without much longer time series. Furthermore, in general, growth rates and saving rates would be jointly determined.

the net saving rate is constant so the resulting gross saving rate decreases in $n$. In the textbook Solow model, the gross saving rate is constant $n$, while the net saving rate increases in $n$. As a result, the empirical evidence aligns best with the model of optimal saving, which implies that both saving rates increase with the rate of population growth. Although through a different path, this evidence points to the same conclusion reached by Krusell and Smith (2015).

## 4. Concluding remarks

Piketty's *Capital in the Twenty-First Century* (2014) has sparked an intense debate inside and outside the profession. It has quickly become one of the most influential books written by academic economists since the turn of the century. Given the considerable amount of public attention this type of non-academic writings of leading scholars attract, it is important to apply rigorous academic standards to clarify the plausibility of their underlying assumptions. In my view, Krusell and Smith (2015) is an excellent example of this. Nonetheless, more often than not, rigor turns out to be a double-edged sword.

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